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COMMENTS ON THE PAPER "MODIFIED GURTIN'S VARIATIONAL PRINCIPLES IN THE LINEAR DYNAMIC THEORY OF VISCOELASTICITY [1]"

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Gurtin[2] was the first to obtain variational principles for initial-value problems, by transforming the governing differential equations and corresponding boundary and initial conditions into equivalent boundary-value problems involving integrodifferential equations, which contained the initial conditions implicitly. In 1974[3] we presented an alternative approach for the formulation of such principles that does not require the transformation of the original problem. On that occasion we mentioned that Gurtin-type variational principles were available for many problems, and showed that for every one of such principles a simplified version could be obtained by taking time derivatives of the corresponding functionals; first or second derivatives, depending on the order of the time operator involved. As an illustration, a simplified principle was obtained for elastodynamics by taking the second time derivative of a functional developed by Gurtin[4].

This simple method can also be applied to the linear dynamic theory of viscoelasticity and, therefore, simplified variational principles can be obtained by taking second order time derivatives of the functionals presented by Leitman[5]. For instance, a slight modification of the results in [5] applicable to the problem defined by eqns (4.1)–(4.3a) and (4.4)–(4.7) of Reddy's paper[1], yields the functional§

$$\Lambda_t = \frac{1}{2}\int_\Omega \{l * G_{ijkl} * \gamma_{ij} * \gamma_{kl} + \rho u_i * u_i - 2g * \sigma_{ij} * \gamma_{ij} - 2(g * \sigma_{ij,j} + b_i) * u_i\}\,\mathrm{d}\mathbf{x}$$
$$+ \int_{\partial\Omega_u} \mathrm{g} * \mathrm{T}_i * \hat{\mathrm{u}}_i\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega_\sigma} g * (T_i - T_i) * u_i\,\mathrm{d}\mathbf{x}. \quad (1)$$

The notation in this equation follows closely that of Reddy, with

$$l(t) = 1, \quad g(t) = t, \quad t \geq 0$$
$$b_i(\mathbf{x}, t) = \rho(\mathbf{x})\{g * f_i(\mathbf{x}, t) + t v_i(\mathbf{x}) + d_i(\mathbf{x})\}, \quad t \geq 0. \quad (2)$$

Thus, according to our results[3] a simpler functional for the same problem is obtained by taking the second order time derivative of eqn (1),

$$J \equiv \ddot{\Lambda}_t = \frac{1}{2}\int_\Omega \{\rho \dot{u}_i * \dot{u}_i + 2\rho(u_i(\mathbf{x}, 0) - d_i(\mathbf{x}))\dot{u}_i(\mathbf{x}, t) - 2\rho v_i(\mathbf{x}) u_i(\mathbf{x}, t)$$
$$+ E_{ijkl}\gamma_{ij} * \gamma_{kl} + \dot{G}_{ijkl} * \gamma_{ij} * \gamma_{kl} - 2\sigma_{ij} * (\gamma_{ij} - u_{i,j}) - 2\rho f_i * u_i\}\,\mathrm{d}\mathbf{x}$$
$$+ \int_{\partial\Omega_u} T_i * (\hat{u}_i - u_i)\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_\sigma} \hat{T}_i * u_i\,\mathrm{d}\mathbf{x}. \quad (3)$$

This is essentially the result obtained by Reddy[1], except for the fact that his eqn (4.20) contains an error, as is explained in the next paragraph. The same procedure can be used to obtain simplified variational principles for the other formulations given by Leitman[5].

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§Equation (4.1) of [5].

Equation (4.13) of [1] reads

$$\left[\frac{\partial f}{\partial t}, g\right] = \left[f, \frac{\partial g}{\partial t}\right] + [f, g]_0 .$$

This equation shows that $\partial/\partial t$ is only formally self-adjoint with respect to the bilinear form [,] because of the presence of the term $[f, g]_0$, and is truly self-adjoint on any subspace of functions f, g for which $[f, g]_0$ vanishes. Whenever operators which are only formally self-adjoint are used to construct variational principles the variations must be taken on subspaces on which the operators are truly self-adjoint. This is usually accomplished by requiring that the admissible functions satisfy certain boundary or initial conditions. Using these facts, it is not difficult to see that the admissible functions in Reddy's principles have to be restricted to satisfy the given initial displacements and velocities. Indeed, this is implicit in the proof of Theorem 4.1, since the variations $\bar{u}_i(\mathbf{x}, 0)$ and $\dot{\bar{u}}_i(\mathbf{x}, 0)$ are taken to be zero. When the functional given by eqn (3) of these comments is used the admissible functions are not restricted in this manner.

Many variational principles have been published in recent years that exhibit limitations similar to the one just discussed. These limitations can be overcome by applying a more systematic formulation of variational principles based on the use of functional valued operators, such as the one we have recently developed [6, 7]. This theory leads to operators which are truly self-adjoint and allows the derivation of veritable extremal principles [6, 7].

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